The Geneva Papers on Risk and Insurance Vol. 29 No. 2 (April 2004) 183–201

Improving Risk Allocation Through Indexed Cat Bonds

by Martin Nell and Andreas Richter*

1. Introduction

Damages inflicted by natural catastrophes in recent years have accounted for economic losses of a size hitherto unknown.[1] The estimated loss potential of some catastrophe scenarios seemingly shows the capacity limits of traditional insurance markets. For instance, estimations of insured losses after a major earthquake in the San Francisco area amount to approximately U.S.$ 100 billion; on the other hand, balance sheets of the U.S. property liability insurance industry show a cumulative surplus of about U.S.$ 300 billion,[2] which, of course, is available not only for catastrophic risks.

These "capacity gaps" in the industry[3] have been at the heart of many discussions among insurance economists and practitioners in the recent past, largely aimed at the development of possible solution strategies involving the financial markets. Contributions can be expected, if, for example, the issuance of marketable insurance-linked securities was able to attract additional capacity from investors who are not otherwise related to the insurance industry. In practice, rudiments of this kind have been observed in various forms since 1992, even though they have yet to reach a significant market share.[4]

To summarize these arguments, the existence of insurance-linked securitization is often explained by its ability to (partly) close the capacity gap of the insurance supply, especially in terms of reinsurance.[5] This line of reasoning is, however, not entirely convincing. Additional risk financing capacity could also be generated through extending capital funds held by the insurance industry or through market entries in the insurance markets. The latter, in fact, could be observed during the 1990s following hurricane Andrew. Immediately after this event reinsurers were very reluctant to cover catastrophe risk and in particular the Lloyd's reinsurance market went through a major crisis, leading to a decline in

* Martin Nell is Professor of Risk and Insurance, University of Hamburg, Germany; Andreas Richter is Assistant Professor of Insurance, Illinois State University, U.S. The authors would like to thank Klaus Bender, Heico Kerkmeester and Howard Kunreuther for very helpful comments on earlier versions of this paper. All errors and omissions remain the responsibility of the authors.

[1] As the dramatic recent events have sadly shown, there are certain risks of man-made disasters that can incur even worse economic and insured losses than extreme natural catastrophes.

[2] See e.g. Cholnoky *et al*. (1998) or Cummins *et al*. (2002).

[3] For an approach measuring the (re)insurance markets' capacity for covering catastrophe risks see Cummins, Doherty and Lo (2002).

[4] See Swiss Re (1999). The total volume of transactions carried out since then exceeds U.S.$ 13 billion (see *Munich Re ART Solutions*, 2001, p. 11). Compared to the size of the reinsurance market, this is still not very significant. For example, the catastrophe excess of loss coverage purchased in the worldwide reinsurance market in the year 2000 amounted to SFr107 billion (Durbin, 2001, p. 301).

[5] See e.g. Kielholz and Durrer (1997), Cholnoky *et al*. (1998).

Published by Blackwell Publishing Ltd, 9600 Garsington Road, Oxford OX4 2DQ, UK.

the supply of catastrophe coverage.[6] Nevertheless, the available reinsurance capacity definitely increased over the next few years as more capital flowed into the industry.[7] As empirical results by Cummins *et al*. (2002) confirm, the insurance industry's capacity as a whole rose between 1991 and 1997. Concurrently, reinsurance prices decreased and the at first rapid increase in the use of alternative risk-financing tools halted in the late 1990s (see Swiss Re, 2001, p. 18).

The terrorist attacks of 11 September 2001 caused another capital and capacity shock for insurance and, in particular, the reinsurance industry.[8] The resulting increase in catastrophe reinsurance prices again provides a framework for a medium-term gain of the insurance-linked securities' market share,[9] as insurers might reconsider the structure of their risk management portfolio. However, higher prices attract more capacity in the catastrophe reinsurance market, which in turn eventually leads to decreasing reinsurance rates.

These examples illustrate that limits to the insurance markets' capacity do not seem to be exogenously given and permanent, but temporary. Therefore, since capacity limits do not provide a convincing rationale for the relevance of insurance-linked securities, we have to consider their special features.[10] In comparison to traditional reinsurance cover, insurance risk securitization has to include elements that provide specific advantages for covering certain risks. They have to be analysed in detail to enhance the understanding of these securities' importance and possible usefulness. Our paper is an endeavour to shed light on the specific advantages and disadvantages of a certain type of insurance securitization, namely the issuance of catastrophe bonds (or cat bonds, for short).

A cat bond is a contract between an issuer and an investor. The investor puts up an amount of cash at the beginning of the coverage period; this is held in escrow until either a pre-specified triggering event occurs or the coverage period ends. The issuer offers a certain coupon payment exceeding the risk-free rate at the end of the period, provided that the event does not occur, and returns both principal and interest to the investor. Otherwise, if the event happens, the investor will receive no coupon payment, and some or all of the principal may go to the issuer.[11]

Three different types of triggering events are usually distinguished. First, the event can be defined directly upon a primary insurer's actual losses. A second possibility is to use

---

[6] See Swiss Re (1998), p. 14.

[7] In particular, reinsurers located in the Bermudas were a major source for additional capacity provided during this period. Companies specialized in natural catastrophe reinsurance were set up and the Bermudas quickly became a very important market. For example, the global market share of the Bermuda reinsurance market developed from 0 per cent to 5 per cent between 1992 and 1997. Apparently, being specialized in natural catastrophe risk, it benefited from increased premiums in this segment and also from relatively lower natural catastrophe losses between 1995 and 1997 (Swiss Re, 1998, pp. 12-21).

[8] See, e.g., Doherty *et al*. (2002).

[9] Terrorism risk itself is an example of a risk category where coverage generated through cat bonds might be an interesting alternative or addition to traditional insurance solutions. See e.g. Kunreuther (2002), who suggests incorporating federal cat bonds as an element of a public-private approach to covering terrorism risk. Sovereign cat bonds have also been discussed in a different problem context: for instance, Croson and Richter (2003) discuss the usefulness of sovereign cat bonds issued by developing countries for the primary purpose of generating conditional funds for infrastructure emergency repairs after catastrophic events.

[10] See also Jaffee and Russel (1997), who argue that the insurance industry's problems in covering catastrophe risks are caused by the institutional framework, since it limits the incentives for holding sufficiently large amounts of liquid capital which would be needed to spread such risks over time.

[11] For the structure of recent cat bonds see Doherty (1997a); for the case of non-indexed cat bonds see also Bantwal and Kunreuther (2000).

aggregate loss data like an industry loss index. Third, the event can be described by a set of technical parameters (parametric trigger), for example, an earthquake's Richter scale reading and location, etc. Cat bonds that link payouts to e.g. an insurer's individual losses are indemnity contracts similar to traditional (re)insurance. In the following we will concentrate on cat bonds with the latter two types of trigger mechanisms which will also be referred to as indexed cat bonds.

Compared to traditional reinsurance, such indexed cat bonds exhibit highly imperfect risk allocation, since they are based on stochastic variables which are not identical with the losses to be covered. To be of any use they have to be correlated with those losses, but usually cannot be a perfect hedge. Thus a buyer of index-linked coverage always has to face the so-called *basis risk*.

On the other hand, index-linked coverage comes with certain attractive features. First, it is a tool to reduce *moral hazard*. While usually in the case of traditional (re)insurance the insured is, more or less, in a position to influence the loss distribution, the pay-off from a cat bond can be based on an underlying stochastic which cannot be controlled or heavily influenced by the buyer.[12] Furthermore, in contrast to catastrophe reinsurance contracts, cat bonds are not subject to *default risk*. While a catastrophic event could influence a reinsurer's ability to compensate the primary, this problem can be avoided by cat bonds: the issuer of a cat bond hedges loss payments without credit risk since his or her obligation to pay interest and/or principal to the investors is forgiven when the bond is triggered.

Another advantage of cat bonds can be seen in the potentially lower prices in comparison to insurance or reinsurance products. Especially in the area of catastrophe reinsurance the premiums entail substantial loadings.[13] According to catastrophe reinsurance data presented by Froot (2001), for example, the average ratio of premiums to expected losses between 1989 and 1998 was higher than four. A very convincing rationale for the high cost of catastrophe reinsurance is the reinsurers' *risk aversion*. Strong evidence for this can be seen in the observation that prices seem to be the greater the higher the reinsurance layers.[14]

Cat bonds, on the other hand, are only weakly correlated with market risk, implying that in perfect financial markets these securities could be traded at a price including just small risk premiums.[15] Furthermore, cat bonds have no or only modest costs of acquisition, monitoring and loss adjustment,[16] which are usually quite considerable in insurance markets. It might be argued, however, that the profitability of insurance-linked securities traded on financial markets so far significantly exceeded the risk free rate. A plausible explanation for this is that high returns were necessary to attract investors to this new kind of

[12] For a detailed discussion of the moral hazard aspect see Doherty and Richter (2002).

[13] See Froot and O'Connell (1999), p. 200, Doherty (1997b), p. 714.

[14] See Froot (1997), p. 5, or Froot (2001), p. 540.

[15] See e.g. Litzenberger *et al*. (1996) or Lewis and Davis (1998). It has to be mentioned, however, that the low correlation argument seems to be of limited validity for major catastrophes. A related objection, when we consider the positive correlation case, is the following: depending on the actual design of a cat bond, for an investor losing money due to the bond being triggered could be a low probability high consequence event implying relatively high risk premiums. This problem, however, is mainly a question of design and in turn depends on the liquidity of the market, the major question being the ratio of potential loss over the invested amount of money. In our model, we will consider the general case, allowing for a positive loading on the expected index-linked coverage.

[16] In particular, in this context one has to mention the advantage of a parametric trigger, that the parameters can be determined very quickly which ensures a rather rapid timing of payments.

transaction. One can expect that, with an increasing degree of standardization and an increasing market share the interest rate of indexed cat bonds will decrease. Moreover, a considerable share of the cat bonds that have been traded in the past has been using triggers defined upon individual actual losses. For these products which are not the focus of this paper, the necessity of monitoring due to moral hazard is a reason for substantially higher prices.[17] Thus, since on the one hand, from a theory point of view, index-linked coverage has the potential to incur only low transaction costs and, if at all, small risk premiums, but on the other hand actual prices significantly exceed the risk-free rate, our model will be quite general. It allows for the cat bond premiums to include a positive loading.

The purpose of this paper is to determine how the availability of coverage generated through the issuance of an index-linked cat bond affects the structure of an optimal indemnity reinsurance contract. For this purpose we consider the case of a primary insurer facing a catastrophic risk that endangers its insured portfolio. The primary can purchase traditional reinsurance as well as index-linked coverage. We consider the optimal mix of both instruments which trades off the reinsurer risk aversion against the basis risk. By comparing optimal (re)insurance indemnity functions for the case with and without the indexed cat bond, we illustrate the impact the introduction of index-linked coverage has on the optimal (re)insurance contract in a standard insurance demand theory framework.

Due to the importance of risk aversion as a determinant of catastrophe reinsurance rates, our analysis will concentrate on this point respectively on the trade-off between a reinsurer's risk aversion and an indexed cat bond's basis risk. In doing so, this paper does not incorporate moral hazard and credit risk as features of the reinsurance contract which, as has been discussed above, can be seen as additional important explanations for a demand in indexed cat bonds. These features would certainly enhance certain structural results that will be derived here.

This paper is related to Doherty and Richter (2002), who also introduce a model to formally address the attractiveness of a joint use of insurance and index-linked coverage. The authors consider the case that insurance can be used to insure the basis risk, which means the policyholder can purchase a separate policy, called gap insurance, to cover the difference between the index-linked coverage and the actual loss.[18] So, the reinsurance contract is, in contrast to our model, defined in a way that the indemnity directly depends on the realization of basis risk. The analysis concentrates mainly on the trade-off between basis risk and moral hazard. Using mean variance to describe the primary's preferences, and assuming a risk neutral reinsurer, it is shown that combining the two hedging tools might extend the possibility set and by that means lead to efficiency gains.

Our paper is organized as follows. In section 2 we introduce our model. The impact of a change in basis risk on the demand for index-linked coverage as well as the reservation price for cat bonds is discussed in section 3, under the assumption that no reinsurance coverage is available. The optimal risk management mix, including the issuance of indexed cat bonds as well as the demand for reinsurance, is the topic of section 4. In section 5 we summarize and discuss our results.

---

[17] See Cummins *et al*. (2004).

[18] The concept of gap insurance for the context of catastrophic risk was first suggested by Croson and Kunreuther (2000).

2. The model

Cat bonds are, as was mentioned above, an imperfect hedging tool, implying basis risk for the primary insurer. It seems suitable to measure the basis risk by the covariance of the primary's actual losses and the index, and then to use a mean variance approach to analyse the demand for index-linked coverage. This kind of analysis, however, has certain disadvantages. First, catastrophe risk, as considered here, is usually represented by considerably skew distributions for which mean variance is problematic. Furthermore, most of the literature on insurance demand theory, which this paper is aimed to link with, is based on expected utility analysis. Therefore, we introduce a different approach to measure basis risk, that enables us to study the interaction between cat bonds and reinsurance in a simple expected utility model.

We consider a risk averse primary insurer[19] that faces stochastic losses X from an insured portfolio.[20] It can purchase index-linked coverage A, which would be triggered with probability $\bar{p}$. Since this kind of product is usually defined discretely, we concentrate, without major loss of generality, on the simple case of a stochastic variable with only the two possible outcomes 0 and A. The primary receives the payment if an exogenous trigger variable Y which is correlated with X reaches a certain level $\bar{y}$.

The correlation between X and Y is expressed by means of the conditional probabilities

$$p(x) := P\{Y \geqslant \bar{y} | X = x\} \tag{1}$$

(if a certain outcome is not specified we also write $p(X)$).

With these definitions clearly: $\bar{p} = E[p(X)]$. $E[\cdot]$ denotes the expectation with regard to the distribution of X.

An intuitive way to develop an idea about adequate assumptions regarding the function $p(x)$ is to look at extreme cases: If, on the one hand, the conditional trigger probability $p(x)$ does not depend on x ($p(x) \equiv \bar{p}$), the index-linked coverage turns out to be completely useless in terms of risk allocation. The primary cannot reduce the risk from its portfolio by issuing a cat bond. So it would simply worsen its situation by buying additional risk.

On the other hand, consider the following problem. Without any further restrictions, construct an index-linked product with two possible outcomes that is optimal in terms of risk allocation. This product would have to be designed in such a way that the payment A is triggered with probability $p(x) = 0$ for losses up to a certain level, but that it is triggered with certainty if X reaches or exceeds this level. This is due to the feature of decreasing marginal utility, which characterizes a risk averse decision-maker's von Neumann-

[19] While risk aversion is widely accepted with respect to the decision behaviour of individuals, firms are often considered as risk neutral. A typical rationale offered for the risk neutrality assumption is that the shareholders hold well-diversified portfolios and will thus aim to maximize the expected profit of the firm (see e.g. Doherty, 1985, p. 465; Shavell, 1987, p. 189; Milgrom and Roberts, 1992, p. 187). However, it is well-known that, due to certain information asymmetries between stockholders and the management, a manager's income should depend on the firm's profit. The individual manager cannot perfectly diversify his or her profit-dependent income. Therefore, even if stockholders act risk neutral because of diversification opportunities, some of the most influential decision-makers will exhibit risk aversion, in particular if they are confronted with the possibility of large losses (see among others Greenwald and Stiglitz, 1990; Dionne and Doherty, 1993; Nell and Richter, 2003).

[20] In the model, the explanation for the primary's demand for hedging tools is its risk aversion. Naturally, there are possible additional motives for corporate demand for hedging tools, such as the wish to avoid or reduce costs of financial distress and bankruptcy costs (see e.g. Mayers and Smith, 1982).

Morgenstern utility function.[21] A situation like this, however, is conceivable only if the coverage can be tied directly to X. But then the product would suffer exactly the same moral hazard problems as traditional reinsurance. Since we want to concentrate on instruments that eliminate especially these problems by connecting the coverage to an exogenous index, the situation mentioned above can just be seen as a limiting case for our analysis.

To be useful as a hedging tool, a cat bond which is based upon an exogenous index must not be completely independent of X. It will, however, in general not be an optimal risk allocation device in the sense defined above. To keep our argument as general as possible, we assume that $p(x)$ vanishes for sufficiently small x, and that $p(x) = 1$ for sufficiently large losses, and finally that there is an area where the trigger probability is strictly between 0 and 1 and increasing. To formalize this, we say that potential levels of loss x_1 and x_2 ($x_1 < x_2$) exist such that[22]

$$\begin{array}{ll} p(x) = 0 & x \leqslant x_1 \\ 0 < p(x) < 1 & x_1 < x < x_2 \\ p(x) = 1 & x \geqslant x_2. \end{array} \tag{2}$$

The intuition behind this is as follows: If the primary is only hit by a very small amount of losses from its portfolio, it is highly unlikely that a triggering event occurred. Given the primary's actual losses are even below x_1, the likelihood that the cat bond was triggered is equal to 0. The conditional trigger probability increases in the amount of actual losses between x_1 and x_2, and finally, our assumptions mean that extremely high individual losses would only be observed if also a triggering event happens. In the model this translates to: given the information that the primary's individual losses exceed x_2, the probability of a triggering event is equal to 1.[23]

As was mentioned above, we assume that $p(x)$ is increasing, which reflects that, *ceteris paribus*, the likelihood of the triggering event, conditioned upon the individual losses, is the greater the higher these individual losses are. The function $p(x)$ is assumed to be differentiable with:

$$p'(x) > 0 \text{ for } x_1 < x < x_2. \tag{3}$$

3. The demand for cat bonds

Before the simultaneous use of both risk management tools is analysed, it makes sense to consider them separately.[24] The optimal demand for reinsurance, on the one hand, has

---

[21] Note that even in this case there would still be a considerable degree of basis risk. For small losses the primary would not receive any pay-off from the cat bond, while in case of losses exceeding the critical level there would in general be a gap between actual losses and the payment A from the index product.

[22] In comparison with the ideal situation mentioned earlier, basis risk is increased, since in addition to the fact that the index-linked coverage is an incomplete fit with respect to the extent of losses, there is, in the interval (x_1, x_2), the uncertainty regarding whether it is triggered or not.

[23] x_2 will usually be related to the triggering level of the index, $\overline{y}$, in the following way: the higher the level of the index has to be to trigger the index-linked coverage, the lower, all other things equal, is the likelihood that, when a given extent of actual losses (x) is observed, the index product is triggered, too. And the higher is the level of actual losses for which the primary can actually be certain to receive the payment from the cat bond.

[24] As has been mentioned above, we consider a situation where reinsurance and index-linked coverage are independent tools. In particular, this means that even when cat bond payments exceed the actual losses, there might still be positive indemnity from the reinsurance contract. Here, our analysis differs from Doherty and Richter (2002).

been discussed extensively in literature.[25] Since, on the other hand, the demand for index-linked coverage has not yet been studied comprehensively in economic models,[26] comparable results are not available for this instrument. The quality of reinsurance as a hedging device is completely determined by the extent of coverage and its price. In the case of indexed cat bonds an additional component has to be taken into account, which is the connection between the stochastic variable the trigger is defined upon, and the individual losses. This connection is expressed by the function $p(x)$ in our model. We will therefore start by analysing the demand for index-linked coverage for different shapes of $p(x)$. We especially want to investigate the impact of a change in basis risk. After that, the question will be addressed, under which conditions a cat bond would be issued at all. Given the degree of basis risk, this leads to determining the price at which the demand for index-linked coverage ceases.

As was mentioned earlier, cat bonds are often considered as only weakly correlated with market risk implying that the price of such securities should include only small risk premiums. But, as market data show, cat bonds which have been issued offered a substantial premium above the risk free rate,[27] probably to attract investors to this new kind of investment. Furthermore, for the few deals that have taken place, transaction costs obviously have been quite high. One can expect, however, that the cost of issuance will significantly decrease as the market gets more experienced in this field and the degree of standardization of such products increases.[28]

For our analysis, we assume that the price of index-linked coverage is $m \cdot \bar{p} \cdot A$ ($m \geqslant 1$), where the proportional loading (m) reflects the issuance costs.

The optimal index-linked coverage in this framework is a solution to the following optimization problem:

$$\max_A E[p(X) \cdot u_1(W_1 - X - m \cdot \bar{p} \cdot A + A) + (1 - p(X)) \cdot u_1(W_1 - X - m \cdot \bar{p} \cdot A)]$$
$$s.t.\ A \geqslant 0, \tag{4}$$

where W_1 is the initial wealth, and u_1 denotes the (three times continuously differentiable) primary's utility function ($u_1' > 0, u_1'' < 0$). As a first order condition for an interior solution we get

$$m \cdot \bar{p} \cdot E[(1 - p(X)) \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A)]$$
$$= (1 - m \cdot \bar{p}) \cdot E[p(X) \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A + A)]. \tag{5}$$

To analyse the impact of a change, namely a reduction, in basis risk on the optimal coverage, we examine the consequences of *ceteris paribus* varying the function $p(x)$ towards the above-mentioned situation where the index-linked coverage can be tied directly to X. We consider a mean preserving transformation of the conditional trigger probability function that shifts the probability weight to higher values of x. More precisely, we keep the

[25] See e.g. Borch (1960), Raviv (1979).
[26] See, however, Cummins and Mahul (2000), and Doherty and Richter (2002) for insurance demand theory approaches to index-linked products.
[27] See e.g. Bantwal and Kunreuther (2000).
[28] According to observers of the market, this point is supported by the fact that the cat bond market has recently been changing towards lower premiums and a greater number of bonds being issued.

unconditional trigger probability $\bar{p}$ constant and consider the effect of replacing $p(x)$ by a function $\tilde{p}(x)$ with the properties (2), (3), and

$$E[\tilde{p}(X)] = E[p(X)] = \bar{p}, \tag{6}$$

and

$$\tilde{p}(x) \leqslant p(x) \; \forall x \leqslant x_3 \text{ and } \tilde{p}(x) \geqslant p(x) \; \forall x \geqslant x_3 \tag{7}$$

for an $x_3 \in (x_1, x_2)$. Conditions (6) and (7) characterize a mean preserving spread with a single crossing property.[29] To exclude trivial cases we assume

$$P\{X = x\colon \tilde{p}(x) \neq p(x)\} > 0. \tag{8}$$

The idea behind this is that in our setting a product with the same unconditional trigger probability, one that is less likely to be triggered for low levels of actual losses but more likely to be triggered for higher losses, means a better fit to the primary's portfolio. Keeping the expected value (or, in other words, the unconditional trigger probability), $\bar{p}$, constant ensures the comparability of different functions $p(x)$: We restrict the following observation to the effect of an actual variation in the risk and exclude the impact of a mere sideways shift of $p(x)$.

Proposition 1
If the optimal amount of index-linked coverage is positive, it will be strictly increased by a reduction in basis risk as defined in (6), (7), and (8).

Proof: see appendix.

All other things being equal, the primary will buy the more index-linked coverage the better it fits for compensating the losses from its original risk, i.e. the better the hedge is.

Next, we want to look at the conditions under which index-linked products are purchased. First, consider the above-mentioned case that in contrast to our assumptions $p(x)$ is independent of x. Such a cat bond cannot be attractive because (for $m \geqslant 1$)

$$\begin{aligned} & m \cdot \bar{p} \cdot (1 - \bar{p}) \cdot E[u_1'(W_1 - X - m \cdot \bar{p} \cdot A)] \\ & \quad > \bar{p} \cdot (1 - m \cdot \bar{p}) \\ & \quad \cdot E[u_1'(W_1 - X - m \cdot \bar{p} \cdot A + A)] \end{aligned} \tag{9}$$

for any $A > 0$, implying $A = 0$. Under these circumstances, a positive amount of index-linked coverage can only be optimal for $m < 1$. The following proposition generalizes this result.

Proposition 2
Under the assumptions of section 2,

$$A > 0 \Leftrightarrow m < E[p(X) \cdot u_1'(W_1 - X)]/(\bar{p} \cdot E[u_1'(W_1 - X)]).$$

Proof: see appendix.

[29] See Rothschild and Stiglitz (1970).

The condition given in Proposition 2 is quite plausible, in particular, since

$$COV[p(X), u_1'(W_1 - X)] = E[p(X) \cdot u_1'(W_1 - X)] - \bar{p} \cdot E[u_1'(W_1 - X)], \qquad (10)$$

it means that a mean-preserving transformation of $p(x)$ which increases the covariance between the conditional trigger probability and the marginal utility in the uninsured situation, enlarges the price the primary is willing to pay for index-linked coverage.

In the following we will assume that the price of index-linked coverage is not prohibitively high, and that it remains attractive even when reinsurance is available as an additional risk management tool. We will therefore assume the optimal amount of coverage from the cat bond to be positive, as we are particularly interested in the impact the index-linked coverage has on optimal reinsurance contracting.

4. The optimal risk management mix

We now turn to the analysis of a simultaneous decision on index-linked coverage and reinsurance. As already mentioned, we assume that the reinsurer is risk averse. The reinsurance premium is denoted by P, $I(x)$ denotes the indemnity function, i.e. $I(x)$ is the amount of indemnity the reinsurer pays if the primary loses x. u_2 is the concave (and three times continuously differentiable) reinsurer's utility function.

As a point of reference, let us first introduce the optimization problem for the case when only reinsurance is available. We derive a Pareto-optimal solution according to:[30]

$$\max_{I(\cdot),P} \alpha \cdot E[u_1(W_1 - X - P + I(X))] + \beta \cdot E[u_2(W_2 + P - I(X))]$$
$$s.t.\ I(x) \geqslant 0\ \forall\ x, \qquad (11)$$

where α and β are positive constants, and W_2 is the reinsurer's initial wealth.

Let $I_0^*(\cdot)$ and P^* denote the solution of (11). As was shown by Raviv (1979), $I_0^*(\cdot)$ has the following properties:[31]

$$I_0^*(0) = 0, \qquad 0 < I_0^*(x) < x,\ \frac{dI_0^*}{dx} > 0 \text{ for } x > 0. \qquad (12)$$

For every loss level, indemnity is strictly positive but incomplete and the coverage strictly increases in x.

Now, consider the optimal risk management mix according to

$$\max_{I(\cdot),A} \alpha \cdot E[p(X) \cdot u_1(W_1 - X - m \cdot \bar{p} \cdot A - P^* + A + I(X))$$
$$+ (1 - p(X)) \cdot u_1(W_1 - X - m \cdot \bar{p} \cdot A - P^* + I(X))] + \beta \cdot E[u_2(W_2 + P^* - I(X))]$$
$$s.t.\ A \geqslant 0,\ I(x) \geqslant 0\ \forall\ x. \qquad (13)$$

Thus, for a moment we keep the reinsurance budget constant in the sense that we assume the same amount of money spent on reinsurance for this case as for the situation with cat bonds, which enables us to concentrate on the implications the availability of index-linked

[30] See e.g. Arrow (1963), pp. 972–973; Raiffa (1970), pp. 196–205; Rees (1985), pp. 21–22.
[31] See Raviv (1979), Theorem 1 (p. 87) and Theorem 3 (p. 90).

coverage has on the *structure* of an ideal reinsurance contract. We will analyse the impact on the budget spent on reinsurance later.

Using the Euler-Lagrange equation the following first-order conditions for an interior solution of problem (13) can be derived:

$$\alpha \cdot p(x) \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + A + I(x)) + \alpha \cdot (1 - p(x)) \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + I(x)) = \beta \cdot u_2'(W_2 + P^* - I(x)) \; \forall x \tag{14}$$

or

$$\frac{p(x) \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + A + I(x)) + (1 - p(x)) \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + I(x))}{u_2'(W_2 + P^* - I(x))} = \frac{\beta}{\alpha} \; \forall x, \tag{15}$$

showing the well-known result, that in a Pareto-optimum the marginal rate of substitution is constant.

Let $I_I^*(\cdot)$ denote the optimal indemnity function from (13). The following results can be derived.

Proposition 3

(a) *For sufficiently large levels of losses x $I_I^*(x)$ is smaller than $I_0^*(x)$. In particular, $p(x) = 1$ implies $I_I^*(x) < I_0^*(x)$. For sufficiently small x, particularly where $p(x) = 0$, $I_I^*(x) > I_0^*(x)$.*

(b) *If the primary is prudent $(u_1''' > 0)$*[32], *$I_I^*(x) > I_0^*(x)$ for x with $p(x) \leqslant m \cdot \bar{p}$.*

Proof: see appendix.

That the optimal reinsurance indemnity for small losses is larger in a situation where index-linked coverage is available, compared to the model without cat bonds, can be explained quite easily: For small x the effect prevails, that the cost of the index-linked product increases the marginal utility of the reinsurance coverage.

To find out more about the optimal indemnity function we consider the slope of $I_I^*(x)$ where $I_I^*(x) > 0$. Applying the implicit function theorem to (14) we get

$$\frac{dI_I^*(x)}{dx} = \frac{\alpha \cdot p(x) \cdot u_1''(W_A) + \alpha \cdot (1 - p(x)) \cdot u_1''(W_B)}{\alpha \cdot p(x) \cdot u_1''(W_A) + \alpha \cdot (1 - p(x)) \cdot u_1''(W_B) + \beta \cdot u_2''(W_C)} - \frac{\alpha \cdot p'(x) \cdot [u_1'(W_A) - u_1'(W_B)]}{\alpha \cdot p(x) \cdot u_1''(W_A) + \alpha \cdot (1 - p(x)) \cdot u_1''(W_B) + \beta \cdot u_2''(W_C)}. \tag{16}$$

(where $W_A := W_1 - x - m \cdot \bar{p} \cdot A - P^* + A + I_I^*(x)$,
$W_B := W_1 - x - m \cdot \bar{p} \cdot A - P^* + I_I^*(x)$, and $W_C := W_2 + P^* - I_I^*(x)$)

The first expression in (16) is positive and smaller than 1, the second is negative only if $A > 0$ and $p'(x) > 0$, otherwise it equals zero. Note that the optimal indemnity function can

[32] The concept of prudence, first proposed by Kimball (1990), is commonly used in insurance demand theory (see e.g. Eeckhoudt and Kimball, 1992; Gollier, 1996). Prudence, e.g. is a necessary condition for constant absolute risk aversion (CARA) respectively decreasing absolute risk aversion (DARA).

be decreasing, especially if the function $p(x)$ is very steep. According to Proposition 3, the optimal policy in the case with cat bonds does not include a deductible. Nevertheless, since the indemnity function can be strictly decreasing, the non-negativity constraint might be binding for some larger level of loss. We will show later that under certain additional conditions this cannot occur (see Proposition 5).

Concerning the comparison of the optimal indemnity functions for the cases with, and respectively without cat bonds, an interesting result can be derived for a certain class of utility functions:

Proposition 4

If the primary's and the reinsurer's preferences are represented by CARA utility functions with risk aversion coefficients a (primary) and b (reinsurer), the slope of the optimal indemnity function in a market with cat bonds is given by

$$\frac{dI_I^*(x)}{dx} = \frac{a}{a+b} - \frac{p'(x) \cdot [1 - e^{-a \cdot A}]}{(a+b) \cdot [p(x) \cdot e^{-a \cdot A} + (1 - p(x))]} \tag{17}$$

as long as $I_I^*(x) > 0$. $I_I^*(x)$ *and* $I_0^*(x)$ *are parallel for* $x \in [0, x_1)$ *and (if* $I_I^*(x) > 0$) *for* $x \in [x_2, \infty)$. *Elsewhere* $I_I^*(x)$ *is less steep than* $I_0^*(x)$.

The connection between the optimal indemnity functions is given by:

$$I_I^*(x) = I_0^*(x) + \frac{a}{a+b} \cdot m \cdot \bar{p} \cdot A + \frac{\ln[p(x) \cdot e^{-a \cdot A} + (1 - p(x))]}{a+b}. \tag{18}$$

Proof: see appendix.

From (18), one can see that, for CARA utility functions, the difference between $I_I^*(x)$ and $I_0^*(x)$ is

$$|I_I^*(x) - I_0^*(x)| = \frac{a}{a+b} \cdot m \cdot \bar{p} \cdot A \tag{19}$$

where $p(x) = 0$, and

$$|I_I^*(x) - I_0^*(x)| = \frac{a}{a+b} \cdot (1 - m \cdot \bar{p}) \cdot A \tag{20}$$

for $p(x) = 1$.

In particular, (19) and (20) imply that, the higher the loading factor for the cat bond premium (m), the greater the increase of reinsurance coverage for very low loss levels, and the smaller the decrease of reinsurance indemnity for very high loss levels, all other things being equal.

Given the reinsurance budget, the existence of catastrophe index-linked securities affects the structure of the reinsurance demand: the index-linked product replaces reinsurance for large losses, whereas the reinsurance indemnity for smaller losses is increased. This confirms the assessment often stated by insurance practitioners, that cat bonds are mainly useful for covering extremely large losses.

So far we have considered the optimal reinsurance contract for the case with index-linked coverage, assuming the premium to be the amount that would be spent in a market without cat bonds. The following proposition now states that the demand for cat bonds reduces the reinsurance budget. The optimal reinsurance contract, given the premium P^*,

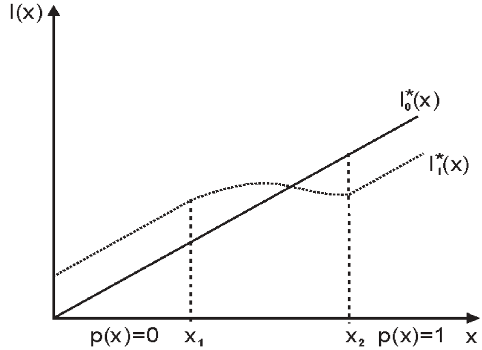

Figure 1

entails a non-stochastic indemnity component. In other words, a certain fraction of that budget would in any case be transferred back to the primary.

Proposition 5
If the primary is prudent and if $p(x) \cdot A \leqslant x$ for all x, the indemnity function $I_I^*(x)$, as derived from (13), includes a positive amount that is paid to the primary with probability one, meaning the de facto reinsurance budget is lower when cat bonds are available, compared to a market without index-linked coverage.

Proof: see appendix.

In the proof it is shown that $I_I^*(0)$ (which, according to Proposition 3, is positive) is the minimum value of the optimal indemnity function. This means that the reinsurance contract can be replaced by a normalized contract with indemnity function $\hat{I}_I^*(x) := I_I^*(x) - I_I^*(0) \geqslant 0$ and premium $\hat{P} := P^* - I_I^*(0)$, leaving the two parties with exactly the same situation as $I_I^*(x)$, P^*. So, under certain conditions, the introduction of cat bonds reduces the budget that is actually spent on reinsurance.

The condition used in the proposition, $p(x) \cdot A \leqslant x$ for all x, can be seen as a purely technical condition in the sense that, if the optimal solution has this property, it is also characterized by the other features stated in the proposition. It could, on the other hand, also be introduced as an additional constraint in the optimization problem (13). This constraint could be interpreted fairly easily. Clearly, $p(x) \cdot A \leqslant x$ is fulfilled in particular whenever the actual loss exceeds the potential cat bond payment. Typical insurance demand theory models incorporate upper bounds that limit the indemnity by the level of loss. But this would obviously not be a useful assumption in our context. Considering the structure of the cat bond, however, it seems to be a straightforward approach to limit the *expected* payment from the bond, given x.

According to the results of this section the introduction of cat bonds has two effects: First, the structure of the optimal reinsurance contract changes. Insurance coverage for large losses decreases while coverage for small losses increases. Second, reinsurance demand, measured as the amount of premium spent on reinsurance, is reduced. Thus, index-linked coverage primarily substitutes reinsurance for large losses, since both effects enhance each other in this area. For lower levels of loss, the structural effect is working in the opposite direction as the wealth of the primary is reduced by the premium for the index-linked product and therefore the marginal utility of reinsurance payments increases.

5. Conclusion

In this paper we consider two important alternatives a primary insurer has for covering catastrophic risks: contracting reinsurance or buying index-linked coverage. We investigate the optimal mix of these instruments. It is shown that there are strong interdependencies, because both means influence each other heavily with respect to their efficiency.

Clearly, the demand for indexed cat bonds can only be explained by imperfections in the reinsurance market, since these bonds always result in a basis risk for the primary insurer. Concentrating on the implicit cost of reinsurance caused by the reinsurer's risk aversion, we analyse the impact of the introduction of catastrophe index-linked coverage on optimal reinsurance. First, we focus on the structure of the optimal contract, keeping the reinsurance budget constant. Under this assumption it is shown that coverage is reduced for large losses and increased for lower losses. A further result is that, under certain conditions, the reinsurance demand in total is reduced: the optimal contract derived for the same budget that would be spent in a market without cat bonds, pays back a certain fraction of the premium to the primary in any case. This means the budget that is *de facto* spent on reinsurance decreases when index-linked coverage is available.

Summarizing these results, our findings indicate that index-linked coverage causes a structural as well as a quantitative effect on the demand for reinsurance. Reinsurance coverage for large losses will be substituted by index-linked coverage, since both effects point in the same direction. However, for small losses the result is ambiguous, since the negative quantitative effect on the demand for reinsurance may be dominated by the positive structural effect.

This paper is, to the best of our knowledge, the first one analysing the interdependencies between the demand for reinsurance and index-linked securities in an expected utility approach. Naturally, many aspects must be left for future research: further insight about the demand for index-linked coverage could probably be derived in a model where the trigger level $\bar{y}$ is an additional decision variable. Moreover, the inclusion of transaction costs might also be an interesting extension, since the influence of insurance-linked securities on the optimal deductible could be examined in that framework. Finally, since reinsurance contracts are subject to substantial problems of moral hazard and default risk, it would be important to take these aspects into account.

REFERENCES

ARROW, K.J., 1963, "Uncertainty and the Welfare Economics of Medical Care", *American Economic Review*, 53, pp. 941–973.

BAMBERG, G. and SPREMANN, K., 1981, "Implications of Constant Risk Aversion", *Zeitschrift für Operations Research*, 25, pp. 205–224.

BANTWAL, V.J. and KUNREUTHER, H.C., 2000, "A Cat Bond Premium Puzzle?", *Journal of Psychology and Financial Markets*, 1, pp. 76–91.

BORCH, K., 1960, "The Safety Loading of Reinsurance Premiums", *Skandinavisk Aktuarietidskrift*, 43, pp. 163–184.

BORCH, K., 1968, "General Equilibrium in the Economics of Uncertainty", in Borch, K. and Mossin, J. (eds), *Risk and Uncertainty.* London: Macmillan, pp. 247–258.

BÜHLMANN, H. and JEWELL, W.S., 1979, "Optimal Risk Exchanges", *Astin Bulletin*, 10, pp. 243–262.

CHOLNOKY, T.V., ZIEF, J.F., WERNER, E.A. and BRADISTILOV, R.S., 1998, "Securitization of Insurance Risk – A New Frontier", Goldman Sachs Investment Research.
CROSON, D.C. and KUNREUTHER, H.C., 2000, "Customizing Indemnity Contracts and Indexed Cat Bonds for Natural Hazard Risks", *Journal of Risk Finance*, 1, pp. 24–41.
CROSON, D.C. and RICHTER, A., 2003, "Sovereign Cat Bonds and Infrastructure Project Financing", *Risk Analysis*, 23, pp. 611–626.
CUMMINS, J.D. and MAHUL, O., 2000, "Managing Catastrophic Risk with Insurance Contracts Subject to Default Risk", Working Paper.
CUMMINS, J.D., DOHERTY, N.A. and LO, A., 2002, "Can Insurers Pay for the 'Big One'? Measuring the Capacity of an Insurance Market to Respond to Catastrophic Losses", *Journal of Banking and Finance*, 26, pp. 557–583.
CUMMINS, J.D., LALONDE, D. and PHILLIPS, R.D., 2004, "The Basis Risk of Catastrophic-Loss Index Securities", *Journal of Financial Economics*, 71, forthcoming.
DIONNE, G., and DOHERTY, N.A. 1993, "Insurance with Undiversifiable Risk: Contract Structure and Organizational Form of Insurance Firms", *Journal of Risk and Uncertainty*, 6, pp. 187–203.
DOHERTY, N.A., 1985, *Corporate Risk Management: A Financial Exposition.* New York: McGraw-Hill.
DOHERTY, N.A., 1997a, "Financial Innovation for Financing and Hedging Catastrophe Risk", in Britton, N.R. and Oliver, J. (eds), *Financial Risk Management for Natural Catastrophes*, Proceedings of a Conference sponsored by Aon Group Australia Limited. Brisbane: Griffith University, pp. 191–209.
DOHERTY, N.A., 1997b, "Innovations in Managing Catastrophe Risk", *Journal of Risk and Insurance*, 64, pp. 713–718.
DOHERTY, N.A. and RICHTER, A., 2002, "Moral Hazard, Basis Risk and Gap Insurance", *Journal of Risk and Insurance*, 69, pp. 9–24.
DOHERTY, N.A., LAMM-TENNANT, J. and STARKS, L., 2002, "Insuring September 11th – Market Recovery and Transparency", Working Paper.
DURBIN, D., 2001, "Managing Natural Catastrophe Risks: The Structure and Dynamics of Reinsurance", *The Geneva Papers on Risk and Insurance*, 26, pp. 297–309.
EECKHOUDT, L. and KIMBALL, M.,1992, "Background Risk, Prudence, and the Demand for Insurance", in Dionne, G. (ed.), *Contributions to Insurance Economics.* Boston: Kluwer Academic Publishers, pp. 239–254.
FROOT, K.A., 1997, "The Limited Financing of Catastrophe Risk: An Overview", NBER Working Paper no. W6025.
FROOT, K.A., 2001, "The Market for Catastrophe Risk: A Clinical Examination", *Journal of Financial Economics*, 60, pp. 529–571.
FROOT, K.A. and O'CONNELL, P.G.J., 1999, "The Pricing of U.S. Catastrophe Reinsurance", in Froot, K.A. (ed.), *The Financing of Catastrophe Risk*. Chicago: University of Chicago Press, pp. 195–231.
GOLLIER, C., 1996, "Optimum Insurance of Approximate Losses", *Journal of Risk and Insurance*, 63, pp. 369–380.
GREENWALD, B.C. and STIGLITZ, J.E., 1990, "Asymmetric Information and the New Theory of the Firm: Financial Constraints and Risk Behavior", *American Economic Review* (Papers and Proceedings), 80, pp. 160–165.
JAFFEE, D.M. and RUSSELL, T., 1997, "Catastrophe Reinsurance, Capital Markets and Uninsurable Risks", *Journal of Risk and Insurance*, 64, pp. 205–230.
KIELHOLZ, W. and DURRER, A., 1997, "Insurance Derivatives and Securitization: New Hedging Perspectives for the US Cat Insurance Market", *The Geneva Papers on Risk and Insurance*, 22, pp. 3–16.
KIMBALL, M.S., 1990, "Precautionary Saving in the Small and in the Large", *Econometrica*, 58, pp. 53–73.
KUNREUTHER, H., 2002, "The Role of Insurance in Managing Extreme Events: Implications for Terrorism Coverage", *Risk Analysis*, 22, pp. 427–437.
LEWIS, C.M. and DAVIS, P.O., 1998,"Capital Market Instruments for Financing Catastrophe Risk: New Directions", *Journal of Insurance Regulation*, 17, pp. 110–133.
LITZENBERGER, R.H., BEAGLEHOLE, D.R. and REYNOLDS, C.E., 1996, "Assessing Catastrophe Reinsurance-Linked Securities as a New Asset Class", *Journal of Portfolio Management*, 23, Special Issue, pp. 76–86.
MAYERS, D. and SMITH, C.W., Jr., 1983, "On the Corporate Demand for Insurance", *Journal of Business*, 55, pp. 281–296.
MILGROM, P. and ROBERTS, J., 1992, *Economics, Organization and Management*. Englewood Cliffs, New Jersey: Prentice Hall.
MUNICH RE ART SOLUTIONS, 2001, *Risk Transfer to the Capital Markets – Using the Capital Markets in Insurance Risk Management*. Munich.

NELL, M. and RICHTER, A., 2003, "The Design of Liability Rules for Highly Risky Activities – Is Strict Liability Superior When Risk Allocation Matters", *International Review of Law & Economics*, 23, pp. 31–47.
PRATT, J.W., 1964, "Risk Aversion in the Small and in the Large", *Econometrica*, 32, pp. 122–136.
RAIFFA, H., 1970, *Decision Analysis*. Reading, Mass.: Addison-Wesley.
RAVIV, A., 1979, "The Design of an Optimal Insurance Policy", *American Economic Review,* 69, pp. 84–96.
REES, R., 1985, "The Theory of Principal and Agent", parts 1 and 2, *Bulletin of Economic Research*, 37, pp. 2–26, and 75–95.
ROTHSCHILD, M. and STIGLITZ, J.E., 1970, "Increasing Risk I: A Definition", *Journal of Economic Theory*, 2, pp. 225–243.
SHAVELL, S., 1987, *Economic Analysis of Accident Law.* Cambridge, Mass.: Harvard University Press.
SWISS RE (ed.), 1998, "The Global Reinsurance Market in the Midst of Consolidation", *Sigma* no. 9/1998 (Zurich).
SWISS RE (ed.), 1999, "Alternative Risk Transfer (ART) for Corporations: A Passing Fashion or Risk Management for the 21st Century", *Sigma* no. 2/1999 (Zurich).
SWISS RE (ed.), 2001, "Capital market innovation in the insurance industry", *Sigma* no. 3/2001 (Zurich).
WILSON, R., 1968, "The Theory of Syndicates", *Econometrica*, 36, pp. 119–132.

Appendix

Proposition 1
If the optimal amount of index-linked coverage is positive, it will be strictly increased by a reduction in basis risk as defined in (6), (7), and (8).

Proof:
(5) can be reformulated as

$$m \cdot \bar{p} \cdot E[u_1'(W_1 - X - m \cdot \bar{p} \cdot A)] \\ = E[p(X) \cdot \{(1 - m \cdot \bar{p}) \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A + A) \\ + m \cdot \bar{p} \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A)\}]. \qquad (21)$$

If – starting from an optimal solution – the trigger probability function is transformed according to (6), (7), and (8), the marginal utility levels for large amounts of losses are weighed more heavily. Since u_1' is strictly decreasing, we get:

$$E[p(X) \cdot \{(1 - m \cdot \bar{p}) \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A + A) + m \cdot \bar{p} \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A)\}] \\ < E[\tilde{p}(X) \cdot \{(1 - m \cdot \bar{p}) \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A + A) \\ + m \cdot \bar{p} \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A)\}] \qquad (22)$$

and therefore

$$m \cdot \bar{p} \cdot E[(1 - \tilde{p}(X)) \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A)] \\ < (1 - m \cdot \bar{p}) \cdot E[\tilde{p}(X) \cdot u_1'(W_1 - X - m \cdot \bar{p} \cdot A + A)]. \qquad (23)$$

In order to fulfil condition (5) again after the variation of ar $p(\cdot)$, A has to be increased.
QED

Proposition 2
Under the assumptions of section 2,
$A > 0 \Leftrightarrow m < E[p(X) \cdot u_1'(W_1 - X)]/(\bar{p} \cdot E[u_1'(W_1 - X)])$.

Proof:
(4) leads to $A = 0$ if and only if

$$m \cdot \bar{p} \cdot E[(1 - p(X)) \cdot u_1'(W_1 - X)] \geqslant (1 - m \cdot \bar{p}) \cdot E[p(X) \cdot u_1'(W_1 - X)]. \tag{24}$$

This can be restated as

$$m \cdot \bar{p} \cdot E[(1 - p(X)) \cdot u_1'(W_1 - X)] + m \cdot \bar{p} \cdot E[p(X) \cdot u_1'(W_1 - X)] \geqslant E[p(X) \cdot u_1'(W_1 - X)] \tag{25}$$

respectively

$$m \cdot \bar{p} \cdot E[u_1'(W_1 - X)] \geqslant E[p(X) \cdot u_1'(W_1 - X)], \tag{26}$$

and finally

$$m \geqslant \frac{E[p(X) \cdot u_1'(W_1 - X)]}{\bar{p} \cdot E[u_1'(W_1 - X)]}. \tag{27}$$

QED

Proposition 3
(a) *For sufficiently large levels of losses x $I_I^*(x)$ is smaller than $I_0^*(x)$. In particular, $p(x) = 1$ implies $I_I^*(x) < I_0^*(x)$. For sufficiently small x, particularly where $p(x) = 0$, $I_I^*(x) > I_0^*(x)$.*
(b) *If the primary is prudent ($u_1''' > 0$), $I_I^*(x) > I_0^*(x)$ for x with $p(x) \leqslant m \cdot \bar{p}$.*

Proof:
I_I^* is defined as an optimal reinsurance indemnity function according to

$$\max_{I(\cdot),A} \alpha \cdot E[p(X) \cdot u_1(W_1 - X - m \cdot \bar{p} \cdot A - P^* + A + I(X)) + (1 - p(X)) \cdot u_1(W_1 - X - m \cdot \bar{p} \cdot A - P^* + I(X))] + \beta \cdot E[u_2(W_2 + P^* - I(X))]$$
$$s.t.\ A \geqslant 0,\ I(x) \geqslant 0\ \forall\ x \tag{28}$$

yielding the first order conditions (14) and (15).

$I_0^*(\cdot)$ is the solution of:

$$\max_{I(\cdot)} \alpha \cdot E[u_1(W_1 - X - P^* + I(X))] + \beta \cdot E[u_2(W_2 + P^* - I(X))]$$
$$s.t.\ I(x) \geqslant 0\ \forall\ x. \tag{29}$$

The optimal indemnity function with regard to (29) is defined by

$$\alpha \cdot u_1'(W_1 - x - P^* + I(x)) = \beta \cdot u_2'(W_2 + P^* - I(x))\ \forall\ x. \tag{30}$$

For a given level of losses x $I_I^*(x)$ will be smaller than $I_0^*(x)$, if the left hand side of (14) is smaller than the left hand side of (30), both evaluated at $I_0^*(x)$. This condition is obviously fulfilled for sufficiently large values of x, respectively $p(x)$.

For values of x with $p(x) = 0$, the left hand side in (14) is

$$\alpha \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + I(x)). \tag{31}$$

This expression (for $A > 0$) exceeds $\alpha \cdot u_1'(W_1 - x - P^* + I(x))$, such that $I_I^*(x) > I_0^*(x)$.

In the case of a prudent primary, we can derive the result $I_I^*(x) > I_0^*(x)$ for x with $p(x) \leqslant m \cdot \bar{p}$ from the convexity of u_1':

$$\begin{aligned}
&\alpha \cdot p(x) \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + A + I(x)) \\
&\quad + \alpha \cdot (1 - p(x)) \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + I(x)) \\
&\quad \geqslant \alpha \cdot u_1'(W_1 - x - m \cdot \bar{p} \cdot A - P^* + p(x) \cdot A + I(x)) \\
&\quad > \alpha \cdot u_1'(W_1 - x - P^* + I(x))
\end{aligned}$$

QED

Proposition 4

If the primary's and the reinsurer's preferences are represented by CARA utility functions with risk aversion coefficients a (primary) and b (reinsurer), the slope of the optimal indemnity function in a market with cat bonds is given by

$$\frac{dI_I^*(x)}{dx} = \frac{a}{a+b} - \frac{p'(x) \cdot [1 - e^{-a \cdot A}]}{(a+b) \cdot [p(x) \cdot e^{-a \cdot A} + (1 - p(x))]} \tag{17}$$

as long as $I_I^*(x) > 0$. $I_I^*(x)$ *and* $I_0^*(x)$ *are parallel for* $x \in [0, x_1)$ *and* $x \in [x_2, \infty)$. *Elsewhere* $I_I^*(x)$ *is less steep than* $I_0^*(x)$.

The connection between the optimal indemnity functions is given by:

$$I_I^*(x) = I_0^*(x) + \frac{a}{a+b} \cdot m \cdot \bar{p} \cdot A + \frac{\ln [p(x) \cdot e^{-a \cdot A} + (1 - p(x))]}{a+b}. \tag{18}$$

Proof:

If index-linked coverage is not available or not attractive, as a well-known result from the theory of optimal risk-sharing we get that $I_0^*(x)$ is a linear function:[33]

$$\frac{dI_0^*}{dx} = \frac{a}{a+b} \tag{33}$$

Now consider again the indemnity function for the situation with index-linked coverage. Dealing with constant absolute risk aversion, we can, without loss of generality, use the utility functions $u_1(W) = -1/a \cdot e^{-a \cdot W}$ and $u_2(W) = -1/b \cdot e^{-b \cdot W}$.[34] For this specific case (16) is of the form

[33] See e.g. Arrow (1963). The fundamental work on the features of Pareto-optimal risk-sharing rules goes back to Borch (1960). See also Wilson (1968), Borch (1968), Raviv (1979), Bühlmann and Jewell (1979).

[34] See e.g. Pratt (1964), Bamberg and Spremann (1981).

$$\frac{dI_I^*(x)}{dx} = \frac{-\alpha \cdot a \cdot e^{-a \cdot I_I^*(x)} \cdot e^{-a \cdot (W_1 - x - m \cdot \bar{p} \cdot A - P^*)} \cdot [p(x) \cdot e^{-a \cdot A} + (1 - p(x))]}{-\alpha \cdot a \cdot e^{-a \cdot I_I^*(x)} \cdot e^{-a \cdot (W_1 - x - m \cdot \bar{p} \cdot A - P^*)} \cdot [p(x) \cdot e^{-a \cdot A} + (1 - p(x))] - \beta \cdot b \cdot e^{b \cdot I_I^*(x)} \cdot e^{-b \cdot (W_2 + P^*)}}$$
$$- \frac{\alpha \cdot p'(x) \cdot e^{-a \cdot I_I^*(x)} \cdot e^{-a \cdot (W_1 - x - m \cdot \bar{p} \cdot A - P^*)} \cdot [e^{-a \cdot A} - 1]}{-\alpha \cdot a \cdot e^{-a \cdot I_I^*(x)} \cdot e^{-a \cdot (W_1 - x - m \cdot \bar{p} \cdot A - P^*)} \cdot [p(x) \cdot e^{-a \cdot A} + (1 - p(x))] - \beta \cdot b \cdot e^{b \cdot I_I^*(x)} \cdot e^{-b \cdot (W_2 + P^*)}}. \quad (34)$$

From (14) follows

$$\alpha \cdot e^{-a \cdot I_I^*(x)} \cdot e^{-a \cdot (W_1 - x - m \cdot \bar{p} \cdot A - P^*)} \cdot [p(x) \cdot e^{-a \cdot A} + (1 - p(x))] = \beta \cdot e^{b \cdot I_I^*(x)} \cdot e^{-b(W_2 + P^*)}, \quad (35)$$

such that (34) can be simplified to

$$\frac{dI_I^*(x)}{dx} = \frac{a}{a+b} - \frac{p'(x) \cdot [1 - e^{-a \cdot A}]}{(a+b) \cdot [p(x) \cdot e^{-a \cdot A} + (1 - p(x))]}. \quad (36)$$

By comparing (33) and (36) we see that $I_I^*(x)$ and $I_0^*(x)$ are parallel if $p'(x)$ vanishes. This is the case for $x \in [0, x_1)$ and $x \in [x_2, \infty)$.

To prove (18), note first that from integrating (36) we get

$$I_I^*(x) = I_0^*(x) + \frac{\ln [p(x) \cdot e^{-a \cdot A} + (1 - p(x))]}{a+b} + const. \quad (37)$$

Since the second expression on the right hand side in (37) equals zero for $x = 0$ and because $I_0^*(0) = 0$, the constant can be found as $I_I^*(0)$. Now, consider conditions (14) and (30). For $x = 0$ and under the assumptions of the proposition, (14) leads to

$$\alpha \cdot e^{-a \cdot (W_1 - P^*)} \cdot e^{-a \cdot (I_I^*(0) - m \cdot \bar{p} \cdot A)} = \beta \cdot e^{-b(W_2 + P^*)} \cdot e^{b \cdot I_I^*(0)}. \quad (38)$$

(30) yields

$$\alpha \cdot e^{-a \cdot (W_1 - P^*)} = \beta \cdot e^{-b(W_2 + P^*)}. \quad (39)$$

Therefore, (38) simplifies to

$$e^{-a \cdot (I_I^*(0) - m \cdot \bar{p} \cdot A)} = e^{b \cdot I_I^*(0)} \quad (40)$$

respectively

$$-a \cdot I_I^*(0) + a \cdot m \cdot \bar{p} \cdot A = b \cdot I_I^*(0) \quad (41)$$

and thus $I_I^*(0) = a \cdot m \cdot \bar{p} \cdot A/(a+b)$.

QED

Proposition 5
If the primary is prudent and if $p(x) \cdot A \leqslant x$ for all x, the indemnity function $I_I^*(x)$, as derived from (13), includes a positive amount that is paid to the primary with probability one, meaning the de facto reinsurance budget is lower when cat bonds are available, compared to a market without index-linked coverage.

Proof:
From proposition 3 we know that $I_0^*(0)$ is positive. In the following we will show that $I_I^*(x) \geqslant I_I^*(0)$ for any x.

Assume there is an $\tilde{x}$ with $I_I^*(\tilde{x}) < I_I^*(0)$. Then, obviously

$$\beta \cdot u_2'(W_2 + P^* - I_I^*(\tilde{x})) < \beta \cdot u_2'(W_2 + P^* - I_I^*(0)). \tag{42}$$

Under the assumptions of the proposition, we see that

$$\begin{aligned}
&\beta \cdot u_2'(W_2 + P^* - I_I^*(\tilde{x})) \\
&= \alpha \cdot p(\tilde{x}) \cdot u_1'(W_1 - \tilde{x} - m \cdot \bar{p} \cdot A - P^* + A + I_I^*(\tilde{x})) \\
&\quad + \alpha \cdot (1 - p(\tilde{x})) \cdot u_1'(W_1 - \tilde{x} - m \cdot \bar{p} \cdot A - P^* + I_I^*(\tilde{x})) \\
&\geqslant \alpha \cdot u_1'(W_1 - \tilde{x} - m \cdot \bar{p} \cdot A + p(\tilde{x}) \cdot A - P^* + I_I^*(\tilde{x})) \\
&\geqslant \alpha \cdot u_1'(W_1 - m \cdot \bar{p} \cdot A + A - P^* + I_I^*(\tilde{x})) \\
&> \alpha \cdot u_1'(W_1 - m \cdot \bar{p} \cdot A + A - P^* + I_I^*(0)) = \beta \cdot u_2'(W_2 + P^* - I_I^*(0))
\end{aligned} \tag{43}$$

which contradicts (42).

QED